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$\mathcal{PT}$-symmetric effective mass Schrödinger equations

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Abstract
We outline a general method of obtaining exact solutions of $\mathcal{PT}$-symmetric Schrödinger equations with a position-dependent effective mass. Using this method, exact solutions of some $\mathcal{PT}$-symmetric potentials have been obtained. We have also discussed the construction of isospectral $\mathcal{PT}$-symmetric potentials.

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1. Introduction

Apart from their intrinsic interest, position-dependent or effective mass Schrödinger equations (EMSEs) have found applications in a number of areas, for example, in the study of electronic properties of semiconductors [1], quantum dots [2], quantum liquids [3]. Also, in recent years various theoretical aspects of EMSEs, for example, exact solvability [4], supersymmetric formulation [5], Lie algebraic approach [6] etc, have been studied widely. It has also been found that such equations appear in very different areas. For example, it has been shown that constant mass Schrödinger equations in curved space and those based on deformed commutation relations can also be interpreted in terms of EMSEs [7]. Also, certain types of time-dependent Schrödinger equations can also be reduced to the effective mass form [8].

On the other hand, non-Hermitian quantum mechanics in general and $\mathcal{PT}$-symmetric ones in particular have drawn much attention [9] since the seminal work by Bender *et al* [10]. One of the reasons behind this surge of interest lies in the fact that $\mathcal{PT}$-symmetric Hamiltonians, in spite of being non-Hermitian, can have real eigenvalues [10]. Furthermore, non-Hermitian quantum mechanics have many applications, for example, in the study of delocalizations [11], population dynamics [12], disordered systems [13]. EMSEs also appear in the context of classical description of certain $\mathcal{PT}$-symmetric models [14]. Here our objective is to examine construction of non-Hermitian potentials within the effective mass formalism. In particular, we shall present a general method of constructing exactly solvable $\mathcal{PT}$-symmetric potentials within the framework of effective mass formalism. Subsequently, we shall use this method to obtain exact solutions of some $\mathcal{PT}$-symmetric potentials. We shall also discuss the construction of isospectral potentials. The organization of the paper is as follows: in section 2, we describe a method of solution of the effective mass Scrödinger equation; in section 3 we obtain exact solutions of some $\mathcal{PT}$-symmetric potentials; in section 4 we discuss the construction of isospectral potentials using different mass functions; in section 5, we discuss the symmetry behaviour of the Schrödinger equations and finally section 6 is devoted to a conclusion.

2. Point transformation approach to the effective mass Schrödinger equation

We note that when the mass depends on the position the kinetic energy can be defined in several ways. The most general form of the Hamiltonian can be written as [15]

$$H = \tfrac{1}{4}(m^{\alpha}(x)pm^{\beta}(x)pm^{\gamma}(x) + m^{\gamma}(x)pm^{\beta}(x)pm^{\alpha}(x)) + V(x) \tag{1}$$

where the parameters α, β and γ are constrained by the relation $\alpha + \beta + \gamma = -1$. There are different forms of the Hamiltonian depending on choices of the parameters. Here we shall work with the most popular form, namely, the BenDaniel–Duke form (corresponding to $\alpha = \gamma = 0, \beta = -1$) [16, 17]. In this case the Hamiltonian is invariant under instantaneous Galilean transformation [17]. The corresponding Schrödinger equation is given by

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2m(x)}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right) + V(x)\psi(x) = E\psi(x) \tag{2}$$

The wavefunction $\psi(x)$ should be continuous at the mass discontinuity and the derivative of the wavefunction should satisfy the following condition:

$$\left.\frac{1}{m(x)}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right|_{-} = \left.\frac{1}{m(x)}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right|_{+}. \tag{3}$$

We recall that a Hamiltonian H is said to be $\mathcal{PT}$-symmetric if [9, 10]

$$\mathcal{PT}H = H\mathcal{PT} \tag{4}$$

where $\mathcal{P}$ is the parity operator and $\mathcal{T}$ is the time reversal operator. Their action on the position and momentum operators are given by

$$\mathcal{P}: x \to -x, \qquad p \to -p, \qquad \mathcal{T}: x \to x, \qquad p \to -p, \qquad i \to -i. \tag{5}$$

Using (5) we find that equation (2) will be $\mathcal{PT}$-symmetric if (we take the mass function to be real)

$$m(x) = m(-x), \qquad V^*(-x) = V(x). \tag{6}$$

We note that equation (2) may be solved in different ways. Here we shall use the method of point canonical transformation [18–20]. To this end, we now perform the following transformation of the wavefunction $\psi(x)$,

$$\psi(x) = [2m(x)]^{\frac{1}{4}}\phi(x) \tag{7}$$

and obtain from equation (2)

$$-\frac{1}{2m(x)}\phi''(x) + \frac{1}{4}\left(\frac{m'(x)}{m^2(x)}\right)\phi'(x) + \left[\frac{7m'^2(x) - 4m(x)m''(x)}{32m^3(x)}\right]\phi(x) + V(x)\phi(x) = E\phi(x) \tag{8}$$

where the prime indicates differentiation with respect to x. Next we make a change of the independent variable defined by

$$\bar{x} = \int^x \sqrt{2m(y)}\,\mathrm{d}y. \tag{9}$$

Using (9) in equation (8) we get

$$-\frac{\mathrm{d}^2\phi(\bar{x})}{\mathrm{d}\bar{x}^2} + \Omega(\bar{x})\phi(\bar{x}) = E\phi(\bar{x}) \tag{10}$$

where for the sake of convenience we have used $\phi(x)|_{x=\bar{x}} = \phi(\bar{x})$ and $\Omega(\bar{x})$ is defined by

$$\Omega(\bar{x}) = V(x=\bar{x}) + \left[\frac{7m'^2(x) - 4m(x)m''(x)}{32m^3(x)}\right]_{x=\bar{x}} = V(x=\bar{x}) + V_1(x=\bar{x}). \tag{11}$$

It is important to note that the change of variable in (9) may not always be invertible or at least not easily invertible. But this does not really pose a problem as far as solvability of (2) is concerned. This is because $\bar{x}$ as a function of x is explicitly known from (9) and if we choose $V(x)$ such that

$$V(x) = V_2(\bar{x}) - V_1(\bar{x}) \tag{12}$$

where $V_2(\bar{x})$ is a solvable $\mathcal{PT}$-symmetric potential then the spectrum of (10) will be known and this in turn will give us the spectrum of equation (2). The corresponding wavefunctions can be obtained using (7). In the next section we illustrate the method with a few examples.

3. Examples

3.1. $\mathcal{PT}$-symmetric Scarf II potential

In order to obtain specific potentials it is now necessary to prescribe the mass function $m(x)$. In the present case we choose the form[1] used in [5]:

$$m(x) = \left(\frac{\alpha + x^2}{1 + x^2}\right)^2, \qquad m(x) = m(-x). \tag{13}$$

Then from (9) we get

$$\bar{x} = \sqrt{2}[x + (\alpha - 1)\tan^{-1} x], \qquad -\infty < \bar{x} < \infty \tag{14}$$

and using (14) in (11) we find

$$V_1(x) = \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[-3x^4 + (2\alpha - 4)x^2 + \alpha]. \tag{15}$$

As we mentioned in the last section, it is now necessary to choose $V_2(\bar{x})$ to be a solvable $\mathcal{PT}$-symmetric potential. Let us first consider the Scarf II potential [21, 22],

$$V_2(\bar{x}) = -\lambda\,\mathrm{sech}^2\,\bar{x} - \mathrm{i}\mu\,\mathrm{sech}\,\bar{x}\tanh\bar{x} \tag{16}$$

where $|\mu| < \lambda + \frac{1}{4}$. The energy spectrum of (16) and the corresponding wavefunctions are well known and are given by [21, 22]

$$E_n = -(n - p - q)^2, \qquad n = 0, 1, 2, \ldots < \frac{s + t - 1}{2} \tag{17}$$

$$\phi_n(\bar{x}) = \frac{\Gamma\left(n - 2p + \frac{1}{2}\right)}{n!\,\Gamma\left(\frac{1}{2} - 2p\right)} z^{-p}(z^*)^{-q} P_n^{-2p-\frac{1}{2},-2q-\frac{1}{2}}(\mathrm{i}\sinh\bar{x}) \tag{18}$$

[1] The choice of the mass function $m(x)$ depends on the particular physical problem involved. Although the results of this paper are valid for any mass function satisfying the condition (6), the particular form (13) is chosen for the sake of integrability.

where $z = \dfrac{1 - \mathrm{i}\sinh \bar{x}}{2}$ and p, q are given by

$$p = -\frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4} + \lambda + \mu} = -\frac{1}{4} \pm \frac{t}{2} \tag{19}$$

$$q = -\frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4} + \lambda - \mu} = -\frac{1}{4} \pm \frac{s}{2}. \tag{20}$$

Then using (12), (15) and (16) the original potential $V(x)$ is found to be

$$V(x) = -\lambda \operatorname{sech}^2[x + (\alpha - 1)\tan^{-1} x] - \mathrm{i}\mu \operatorname{sech}[x + (\alpha - 1)\tan^{-1} x] \times \tanh[x + (\alpha - 1)\tan^{-1} x] + \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[3x^4 + (4 - 2\alpha)x^2 - \alpha]. \tag{21}$$

The corresponding wavefunctions can be obtained using (7) and (9) and are given by

$$\psi_n(x) = N_n\sqrt{\frac{\alpha + x^2}{1 + x^2}}\phi_n(\bar{x}) \tag{22}$$

where N_n is a normalization constant and $\bar{x}$ is given by (14).

It can be readily verified that

$$V(x) = V^*(-x) \tag{23}$$

so that the potential (21) is $\mathcal{PT}$-symmetric and has a real spectrum given by (17). Also, from (14) and (18) it follows that the wavefunctions are $\mathcal{PT}$-symmetric.

3.2. $\mathcal{PT}$-symmetric oscillator

Here we shall consider a different type of $\mathcal{PT}$-symmetric problem. We note that the previous example was non-Hermitian because of the presence of a complex coupling constant. In the present case, we shall consider a model where non-Hermiticity enters through a complex shift of coordinates. Keeping the same choice of the mass function as before, we now choose $V_2(\bar{x})$ to be the $\mathcal{PT}$-symmetric generalized oscillator [23]:

$$V_2(\bar{x}) = (\bar{x} - \mathrm{i}\epsilon)^2 + \frac{g^2 - \frac{1}{4}}{(\bar{x} - \mathrm{i}\epsilon)^2}. \tag{24}$$

The energy eigenvalues and the corresponding wavefunctions are given by [23]

$$E_n = 4n - 2qg + 2, \qquad n = 0, 1, 2, \ldots \tag{25}$$

$$\phi_n(\bar{x}) = \mathrm{e}^{-\frac{1}{2}(\bar{x} - \mathrm{i}\epsilon)^2}(\bar{x} - \mathrm{i}\epsilon)^{-qg + \frac{1}{2}} L_n^{-qg}((\bar{x} - \mathrm{i}\epsilon)^2) \tag{26}$$

where $q = \pm 1$ is called the quasi-parity.

Now proceeding as in the last example the effective mass potential and the wavefunctions are found to be

$$V(x) = [x - (\alpha - 1)\tan^{-1} x - \mathrm{i}\epsilon]^2 + \frac{g^2 - 1}{[x - (\alpha - 1)\tan^{-1} x - \mathrm{i}\epsilon]^2} + \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[3x^4 + (4 - 2\alpha)x^2 - \alpha] \tag{27}$$

$$\psi_n(x) = N_n\sqrt{\frac{\alpha + x^2}{1 + x^2}}\phi_n(\bar{x}). \tag{28}$$

As before it is readily seen that the effective mass potential (27) as well as the wavefunctions (28) are again $\mathcal{PT}$-symmetric and (27) has real eigenvalues given by (25).

4. Isospectral $\mathcal{PT}$-symmetric potentials

In the case of a constant mass Schrödinger equation isospectral potentials (both Hermitian [24] and non-Hermitian [25]) have been widely studied using different techniques, for example, the Darboux transformation. However, in the case of an EMSE isospectral potentials can be obtained, for example, with a change of the mass function also. Here we shall use this method to construct isospectral $\mathcal{PT}$-symmetric potentials in the case of an EMSE.

To start with let us consider a function of the form

$$m(x) = \left(\frac{\alpha + x^2}{1 + x^2}\right)^4. \tag{29}$$

Then from (7) we get

$$\bar{\bar{x}} = \frac{1}{\sqrt{2}}\left[2x + \frac{(\alpha-1)^2 x}{(1+x^2)} + (\alpha-1)(\alpha+3)\tan^{-1} x\right], \qquad -\infty < \bar{\bar{x}} < \infty \tag{30}$$

while (11) gives us

$$V_1(x) = \frac{(\alpha-1)(1+x^2)^2}{(\alpha+x^2)^6}[-3x^4 + (5\alpha - 7)x^2 + \alpha]. \tag{31}$$

For $V_2(\bar{\bar{x}})$ let us now choose

$$V_2(\bar{\bar{x}}) = (\bar{\bar{x}} - \mathrm{i}\epsilon)^2 + \frac{g^2 - \frac{1}{4}}{(\bar{\bar{x}} - \mathrm{i}\epsilon)^2} \tag{32}$$

so that its spectrum is given by (25). In this case $V(x)$ is given by

$$\begin{aligned} V(x) = {} & \left[\sqrt{2}x + \frac{(\alpha-1)^2 x}{\sqrt{2}(1+x^2)} + \frac{(\alpha-1)(\alpha-3)}{\sqrt{2}}\tan^{-1} x - \mathrm{i}\epsilon\right]^2 \\ & + \frac{g^2 - \frac{1}{4}}{\left[\sqrt{2}x + \frac{(\alpha-1)^2 x}{\sqrt{2}(1+x^2)} + \frac{(\alpha-1)(\alpha-3)}{\sqrt{2}}\tan^{-1} x - \mathrm{i}\epsilon\right]^2} \\ & + \frac{(\alpha-1)(1+x^2)^2}{(\alpha+x^2)^6}[3x^4 + (7 - 5\alpha)x^2 - \alpha]. \end{aligned} \tag{33}$$

The wavefunctions can be obtained using (7), (26) and (29) and they are of the form

$$\psi_n(x) = N_n \sqrt{\frac{\alpha + x^2}{1 + x^2}}\phi_n(\bar{\bar{x}}) \tag{34}$$

where $\phi_n(\bar{\bar{x}})$ and $\bar{\bar{x}}$ are given, respectively, by (26) and (30). It may be easily checked that the potential (33) as well as the wavefunctions are both $\mathcal{PT}$-symmetric and (33) has exactly the same spectrum as the potential (27). Thus two EMSEs having different mass functions and potentials are isospectral.

5. Symmetry of the potentials

We have seen that the method of solution is based on mapping the EMSE to an exactly solvable constant mass Schrödinger equation. Subsequently, choosing an exactly solvable $\mathcal{PT}$-symmetric potential we obtain an exactly solvable $\mathcal{PT}$-symmetric potential in the EMSE formalism. Here the question is why do the Schrödinger equations (2) and (10) share the same symmetry? In other words, why does $\mathcal{PT}$ symmetry survive the reverse transformation from the constant mass Schrödinger equation to the EMSE? To answer this question, we recall that the potential $V(x)$ is of the form

$$V(x) = V_2(\bar{x}) - V_1(\bar{x}) \tag{35}$$

where $\bar{x}$ on the right-hand side is given by (7). Now $V_2(\bar{x})$ is $\mathcal{PT}$-symmetric. So $V_2(x)$ is $\mathcal{PT}$-symmetric if $\bar{x}$ have the same behaviour as x, i.e,

$$\mathcal{PT}\bar{x} = -\bar{x}. \tag{36}$$

This, in turn, depends on the form of $m(x)$. Indeed from equation (9) it follows that for both the choices of $m(x)$ (i.e, (13) and (29)), equation (36) holds. Also $V_1(x)$ given by

$$V_1(x) = \left[\frac{7m'^2(x) - 4m(x)m''(x)}{32m^3(x)}\right] \tag{37}$$

is $\mathcal{PT}$-symmetric for $m(x)$ satisfying (6). As a consequence $V(x)$ is always $\mathcal{PT}$-symmetric. We now consider the wavefunctions. Since $\phi_n(\bar{x})$ are $\mathcal{PT}$ invariant, it follows from (36) that $\psi_n(x)$ are also $\mathcal{PT}$ invariant. The same conclusion holds in the case of the potential (33) also since $\bar{\bar{x}}$ and x have the same behaviour with respect to $\mathcal{PT}$ transformation.

6. Conclusion

Here we have described a general method of obtaining exact solutions of $\mathcal{PT}$-symmetric effective mass Schrödinger equations. In particular, the method has been applied to obtain solutions of effective mass analogues of $\mathcal{PT}$-symmetric Scarf II and oscillator potentials which have so far been unknown. It is also clear from the procedure that exactly solvable complex non-$\mathcal{PT}$-symmetric potentials can also be constructed. For example, if we take $V_2(\bar{x})$ as the complex Morse potential [26], then we would end up with a non-$\mathcal{PT}$-symmetric $V(x)$ with a real spectrum. We have also examined the construction of isospectral $\mathcal{PT}$-symmetric potentials using a different mass function $m(x)$. This aspect of EMSE is different from the constant mass case. In this context, we would like to discuss the choice of the mass function. The choice of the mass function essentially depends on the specific application. However, the objective of this paper is of an exploratory nature and consequently we have chosen the mass functions (13) and (29) suitable for integrability of (9). Finally, we feel it would be interesting to examine the possibility of constructing effective mass $\mathcal{PT}$-symmetric potentials using other techniques, for example, Lie algebraic methods, supersymmetry, Darboux transformation etc.

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Note added in proof. Recently we have come across two papers by L Jiang *et al* (2005 *Phys. Lett.* A **345** 279) and B Bagchi *et al* (2005 *J. Phys. A: Math. Gen.* **36** L647) which have some overlap with the present paper.

References

[1] Bastard G 1988 *Wave Mechanics Applied to Semiconductor Heterostructures* (Les Ulis: Les Editions de Physique)
[2] Serra L I and Lipparini E 1997 *Europhys. Lett.* **40** 667
[3] Arias de Saavedra F, Boronat J, Polls A and Fabrocini S 1994 *Phys. Rev.* B **50** 4282

[4] Dekar L, Chetouani L and Hammann T F 1999 *Phys. Rev.* A **59** 107
Dekar L, Chetouani L and Hammann T F 1998 *J. Math. Phys.* **39** 2551
de Souza Dutra A and Almeida C A S 2000 *Phys. Lett.* A **275** 25
Milanović V and Iković Z 1999 *J. Phys. A: Math. Gen.* **32** 7001
Koc R, Koca M and Körchük E 2002 *J. Phys. A: Math. Gen.* **35** L527
Chen G and Chen Z 2004 *Phys. Lett.* A **331** 312
Yu J and Dong S H 2004 *Phys. Lett.* A **325** 194
[5] Plastino A R, Rigo A, Casas M, Gracias F and Plastino A 1999 *Phys. Rev.* A **60** 4398
Gönul B, Gönul B, Tutcu D and Özer O 2002 *Mod. Phys. Lett.* A **17** 2057
[6] Roy B and Roy P 2002 *J. Phys. A: Math. Gen.* **35** 3961
Koc R and Koca M 2003 *J. Phys. A: Math. Gen.* **36** 8105
[7] Quesne C and Tkachuk V M 2004 *J. Phys. A: Math. Gen.* **37** 4267
Bagchi B, Banerjee A, Quesne C and Tkachuk V M 2005 *J. Phys. A: Math. Gen.* **38** 2929
[8] Zhang S and Li F 1996 *J. Phys. A: Math. Gen.* **29** 6143
[9] Bender C M, Boettcher S, Jones H F, Meisinger P N and Simsek M 2001 *Phys. Lett.* A **291** 197
Znojil M 2000 *J. Phys. A: Math. Gen.* **33** 4561
Znojil M 2002 *J. Phys. A: Math. Gen.* **35** 2341
Lévai G and Znojil M 2000 *J. Phys. A: Math. Gen.* **33** 7165
Bagchi B and Quesne C 2002 *Phys. Lett.* A **300** 18
Dorey P, Dunning C and Tateo R 2001 *J. Phys. A: Math. Gen.* **34** 5679
Andrianov A A, Cannata F, Dedonder J P and Ioffe M V 1999 *Int. J. Mod. Phys.* A **14** 2675
Mostafazadeh A 2002 *Nucl. Phys.* B **640** 419
Mostafazadeh A 2003 *J. Math. Phys.* **44** 974
Tkachuk V M and Fityo T V 2001 *J. Phys. A: Math. Gen.* **34** 8673
[10] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
Bender C M and Boettcher S 1998 *J. Phys. A: Math. Gen.* **31** L273
[11] Moiseyev N and Glück M 2002 *Phys. Rev.* E **63** 041103
[12] Dahmen K, Nelson D R and Shnerb N M 1998 *Phys. Rev.* E **58** 1383
[13] Hatano N and Nelson D R 1998 *Phys. Rev.* B **58** 8384
Hatano N and Nelson D R 1997 *Phys. Rev.* B **56** 8651
Hatano N and Nelson D R 1996 *Phys. Rev. Lett.* **77** 570 (*Preprint* cond-mat/9801283)
[14] Mostafazadeh A 2005 *J. Phys. A: Math. Gen.* **38** 6557
Mostafazadeh A 2005 *J. Phys. A: Math. Gen.* **38** 8185
[15] Von Roos O 1983 *Phys. Rev.* B **27** 7547
[16] BenDaniel D J and Duke C B 1966 *Phys. Rev.* **152** 683
[17] Lévy-Leblond J-M 1995 *Phys. Rev.* A **52** 1845
[18] Alhaidary A D 2002 *Phys. Rev.* A **66** 042116
[19] Roy B and Roy P 2001 *Preprint* quant-ph/0106028
[20] Gönul B, Özer O, Gönül B and Üzgün F 2002 *Mod. Phys. Lett.* A **17** 2453
[21] Ahmed Z 2001 *Phys. Lett.* A **282** 343
[22] Sinha A and Roychoudhury R 2002 *Phys. Lett.* A **301** 163
[23] Znojil M 1999 *Phys. Lett.* A **259** 220 and references therein
[24] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[25] Sinha A and Roy P 2004 *J. Phys. A: Math. Gen.* **37** 2059
[26] Ahmed Z 2001 *Phys. Lett.* A **290** 19
Bagchi B and Quesne C 2000 *Phys. Lett.* A **273** 285